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THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCI...

OFFICE OF THE REGISTRAR
5-FEB 1925

EDITED BY

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Volume XV, No. 12.

December 1924.

MADRAS:

PRINTED BY SRINIVASA VARADACHARI & CO.

Annual Subscription, Rs. 6.] [Single Copy, One Rupee.

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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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Then $c + is = 1 + e^{i\theta} + e^{2i\theta} + \ldots\ldots\ldots\ldots$

$$+ e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots\ldots\ldots\ldots$$

$$+ e^{2i\theta} + e^{3i\theta} + e^{4i\theta} + \ldots\ldots\ldots\ldots$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The associated function $u(x, y)$ is found to be $e^{(x+y)e^{i\theta}}$.

$$\therefore \quad c + is = \int_0^\infty \int_0^\infty e^{-x(1-e^{i\theta})-y(1-e^{i\theta})}\,dx\,dy = \frac{1}{(1-e^{i\theta})^2},$$

$$= \frac{1}{2} - \frac{1}{4}\operatorname{cosec}^2\frac{\theta}{2} - \frac{i}{2}\cot\frac{\theta}{2},$$

if $0 < \theta < 2\pi$.

Equating the real and imaginary parts, we have

$$c = \frac{1}{2} - \frac{1}{4}\operatorname{cosec}^2\frac{\theta}{2}, \text{ and } s = -\frac{1}{2}\cot\frac{\theta}{2},$$

if $0 < \theta < 2\pi$.

(2) Changing θ to $(\pi - \theta)$ in the double cosine series and the double sine series of the previous example, we have

$$\frac{1}{2} - \tfrac{1}{4}\sec^2\frac{\theta}{2} = 1 - \cos\theta + \cos 2\theta - \cos 3\theta + \ldots\ldots\ldots$$

$$- \cos\theta + \cos 2\theta - \cos 3\theta + + \ldots\ldots$$

$$+ \cos 2\theta - \cos 3\theta + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

provided that $-\pi < \theta < +\pi$;

and $-\tfrac{1}{2}\tan\dfrac{\theta}{2} = 0 + \sin\theta - \sin 2\theta + \sin 3\theta - \ldots\ldots\ldots$

$$+ \sin\theta - \sin 2\theta + \sin 3\theta - \ldots\ldots\ldots$$

$$- \sin 2\theta + \sin 3\theta - \sin 4\theta + \ldots\ldots$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

provided that $-\pi < \theta < +\pi$.

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(3) Again let

$$\begin{aligned} c = \cos\theta &+ \cos 3\theta + \cos 5\theta + \ldots\ldots \\ &+ \cos 3\theta + \cos 5\theta + \cos 7\theta + \ldots\ldots \\ &+ \cos 5\theta + \cos 7\theta + \cos 9\theta + \ldots\ldots \\ &+ \ldots\ldots\ldots\ldots \end{aligned}$$

and

$$\begin{aligned} s = \sin\theta &+ \sin 3\theta + \sin 5\theta + \ldots\ldots \\ &+ \sin 3\theta + \sin 5\theta + \sin 7\theta + \ldots\ldots \\ &+ \sin 5\theta + \sin 7\theta + \sin 9\theta + \ldots\ldots \\ &+ \ldots\ldots\ldots\ldots \end{aligned}$$

As in example (1), it can be easily obtained that

$$c + i\,s = \frac{e^{i\theta}}{\left(1 - e^{2i\theta}\right)^2}$$

$$= \frac{e^{-i\theta}}{\left(e^{i\theta} - e^{-i\theta}\right)^2} = \frac{\cos\theta - i\sin\theta}{-4\sin^2\theta}.$$

Equating the real and imaginary parts, we have

$$c = -\frac{\cot\theta \operatorname{cosec}\theta}{4} \quad \text{and} \quad s = \frac{\operatorname{cosec}\theta}{4},$$

if $0 < \theta < \pi$.

(4) Changing θ into $\frac{\pi}{2} - \theta$, we have

$$\begin{aligned} \frac{\sec\theta}{4} = \cos\theta &- \cos 3\theta + \cos 5\theta - \ldots\ldots \\ &- \cos 3\theta + \cos 5\theta - \cos 7\theta + \ldots\ldots \\ &+ \cos 5\theta + \cos 7\theta + \cos 9\theta + \ldots\ldots \\ &- \ldots\ldots + \ldots\ldots\ldots\ldots \end{aligned}$$

if $-\frac{\pi}{2} < \theta < +\frac{\pi}{2}$;

and

$$\begin{aligned} -\tfrac{1}{4}\,\frac{\sec\theta\tan\theta}{4} = \sin\theta &- \sin 3\theta + \sin 5\theta + \ldots\ldots \\ &- \sin 3\theta + \sin 5\theta - \sin 7\theta + \ldots\ldots \\ &+ \sin 5\theta - \sin 7\theta + \sin 9\theta - \ldots\ldots \\ &- \ldots\ldots + \ldots\ldots\ldots\ldots \end{aligned}$$

if $-\frac{\pi}{2} < \theta < +\frac{\pi}{2}$.

Some Applications of Modular Equations.

By M. VENKATARAMA AYYAR.

§ 1. In the paper on "Some Infinite Products and Series" which appeared in the *J.I.M.S.* (Vol. XV, Nos. 7, 10 and 11), certain results relating to 'Theta-Products'* and 'Inverse Tangent Series' of the type $\Sigma_1 - \Sigma_2$† have been established by Mr. Bhimasena Rao and myself. In this paper, I propose to evaluate, by a direct application of a few modular equations, some fresh Theta-Products and also to discuss the sums of the series Σ_1 and Σ_2 separately.

The modular equations made use of in this paper are those of the second, third and fifth orders,‡ *viz*:—

$$\sin\beta = \frac{1-\cos\alpha}{1+\cos\alpha} \quad \ldots \quad \ldots \quad (a)$$

$$(\sin\alpha\,\sin\beta)^{\frac{1}{2}} + (\cos\alpha\,\cos\beta)^{\frac{1}{2}} = 1, \quad \ldots \quad (b)$$

and $$\left\{(\sin\alpha)^{\frac{1}{2}} - (\sin\beta)^{\frac{1}{2}}\right\}^6 = 16(\sin\alpha\,\sin\beta)^{\frac{1}{2}}(\cos\alpha\cos\beta)^2 \quad \ldots \quad (c)$$

It may be of interest to note that these modular equations may be derived by simple substitutions in some of the identities given on pages 107 and 93 of Bromwich's *Introduction to Infinite Series*. With Q_0, Q_1, Q_2, Q_3 having the meanings given to them on p. 105 of that book, and with the relations

$$Q_1 Q_2 Q_3 = 1, \text{ §} \quad \ldots \quad \ldots \quad (d)$$

and $$Q_2^{\,8} = Q_3^{\,8} + 16 q Q_1^{\,8}, \text{ ¶} \quad \ldots \quad \ldots \quad (e)$$

---

* *Vide J. I. M. S.*, Vol. XV, No. 7, p. 162, Appendix.

† *Vide J. I. M. S.*, Vol. XV, No. 11, Results VII to XIV.

‡ *Vide* Cayley: *Elliptic Functions*, Art. 244, Art. 261 or 271 (II) and Art. 273 (II).

§ *Vide* Bromwich: *Infinite Series* p. 105 Ex. 14, or Cayley's *Elliptic Functions*, Art. 377.

¶ *Vide* Bromwich; *Infinite Series* p. 107 Ex. 20, or Cayley's *Elliptic Functions*, Art. 377.

putting $\left(\frac{Q_3}{Q_2}\right)^4 = \cos \alpha$, it is easily seen that

$$\left.\begin{aligned} Q_2 &= 2^{\frac{1}{4}} q^{\frac{1}{24}} (\sin 2\alpha)^{-\frac{1}{12}}, \\ Q_3 &= 2^{\frac{1}{4}} q^{\frac{1}{24}} (\sin 2\alpha)^{-\frac{1}{12}} (\cos \alpha)^{\frac{1}{4}}, \\ \text{and} \quad Q_1 &= 2^{-\frac{1}{2}} q^{-\frac{1}{12}} (\sin 2\alpha)^{\frac{1}{6}} (\cos \alpha)^{-\frac{1}{4}}. \end{aligned}\right\} \quad \ldots \text{(A)}$$

Using these relations (A) in the identity given under Ex. 20 on p. 107 of Bromwich, we can, by putting $x = i$ and $x = w$, get, on simplification, the modular equations of the second and third orders.

As a sample of this method, we shall just briefly indicate how the modular equation of the fifth order is arrived at by substitution.

On p. 93 of Bromwich's *Infinite Series*, it has been shown that

$$f(\sqrt{xy}, \sqrt{q}) . f\left(\sqrt{\frac{x}{y}}, \sqrt{q}\right) = f(x, q)\, f(y, q) + g(x, q)\, g(y, q)$$

$$\text{and } f(-\sqrt{xy}, \sqrt{q}) . f\left(-\sqrt{\frac{x}{y}}, \sqrt{q}\right) = f(x, q)\, f(y, q) - g(x, q)\, g(y, q).$$

where $\quad f(p, q) = Q_0 . \prod_1^{\infty} (1 + q^{2n-1} p) \left(1 + \frac{q^{2n-1}}{p}\right)$

and $\quad g(p, q) = q^{\frac{1}{4}} p^{-\frac{1}{2}} f\left(\frac{p}{q}, q\right)$.

Multiplying up, substituting for 'f' and 'g,' and simplifying

$$Q_3^4 . \prod_1^{\infty} \left\{ \left(1 - q^{2n-1} xy\right) \left(1 - \frac{q^{2n-1}}{xy}\right) \left(1 - q^{2n-1} \frac{x}{y}\right) \left(1 - q^{2n-1} \frac{y}{x}\right) \right\}$$

$$= \left[\prod_1^{\infty} \left\{ \left(1 + q^{2n-1} x\right) \left(1 + \frac{q^{2n-1}}{x}\right) \left(1 + q^{2n-1} y\right) \left(1 + q^{2n-1} \frac{1}{y}\right) \right\} \right]^2$$

$$- q\,(x^{\frac{1}{2}} + x^{-\frac{1}{2}})^2 (y^{\frac{1}{2}} + y^{-\frac{1}{2}})^2 \times$$

$$\left[\prod_1^{\infty} \left\{ \left(1 + q^{2n} x\right) \left(1 + \frac{q^{2n}}{x}\right) \left(1 + \frac{q^{2n}}{y}\right) \left(1 + q^{2n} y\right) \right\} \right]^2. \quad \text{(B)}$$

In (B), put $x = w$ (a fifth root of unity) and $y = w^3$. Then, denoting by Q', the infinite product corresponding to Q when q^5 replaces q, we

have

$$Q_3^4 . \left(\frac{Q_3'}{Q_3}\right) = \left(\frac{Q_2'}{Q_2}\right)^2 - q\left(\frac{Q_1'}{Q_1}\right)^2$$

or

$$Q_3Q_3' = Q_1^2Q_2'^2 - qQ_2^2Q_1'^2 \ldots \qquad \ldots \text{(C)}$$

If β now denote the modular angle for Q', on substituting from (A) and making a few reductions, we get

$$4\cos\alpha\cos\beta\,(\sin 2\alpha \sin 2\beta)^{\frac{1}{3}} = \{(\sin\alpha)^{\frac{1}{2}} - (\sin\beta)^{\frac{1}{2}}\}^4. \quad \ldots \text{(1)}$$

This is easily seen to be the same as the form in Cayley for the modular equation of the fifth order given in (C). It is also identical with Russel's form

$$\cos(\alpha - \beta) + 2\{\sin 2\alpha \sin 2\beta\}^{\frac{1}{3}} = 1;$$

for, on putting $(\sin 2\alpha \sin 2\beta)^{\frac{1}{3}} = p^2$, (1) becomes

$$(\sqrt{\sin\alpha} - \sqrt{\sin\beta})^2 = 2p . \sqrt{\cos\alpha\cos\beta};$$

which gives $\sin\alpha + \sin\beta = 2(p . \sqrt{\cos\alpha\cos\beta} + \sqrt{\sin\alpha\sin\beta})$.

Squaring and simplifying, we get

$$2p^2 = 1 - \cos(\alpha - \beta)$$

which is Russel's form.

This fifth order modular equation may be put into yet another convenient form which we use below:

$$(\sin 2\alpha \sin 2\beta)^{\frac{1}{3}} = \sin^2\left(\frac{\alpha - \beta}{2}\right).$$

$$\therefore \quad \cos 2(\alpha - \beta) - \cos 2(\alpha + \beta) = 2\sin^6\frac{\alpha - \beta}{2}.$$

$$\therefore \quad \sin^2(\alpha + \beta) = \sin^2\frac{\alpha - \beta}{2}\left\{\sin^4\frac{\alpha - \beta}{2} + 4\cos^2\frac{\alpha - \beta}{2}\right\}.$$

$$\therefore \quad \sin(\alpha + \beta) = \sin\frac{\alpha - \beta}{2}\left(2 - \sin^2\frac{\alpha - \beta}{2}\right). \quad \ldots \text{(2)}$$

§ 2. It was shown that, when

$$q = e^{-\pi}, \; Q_2 = 2^{\frac{1}{4}} q^{\frac{1}{24}} *,$$

such that, from (A), $(\sin 2\alpha)^{-\frac{1}{12}} = 1$

* *Vide J. I. M. S.*, Vol. XV, No. 7, p. 162, Appendix.

giving the modular angle $\alpha = 45°$. We shall now find the values of Q_1, Q_2, Q_3 when $e^{-\pi}$ is changed into $e^{-2\pi}$ and $e^{-3\pi}$. In the first case, we have from (*a*),

$$\sin\beta = (\sqrt{2}-1)^2$$

making
$$\cos\beta = 2^{\frac{5}{4}}(\sqrt{2}-1),$$

and
$$\sin 2\beta = 2^{\frac{9}{4}}(\sqrt{2}-1)^3.$$

$$\text{Hence, } Q_2' = \prod_1^\infty \{1 + e^{-2(2n-1)\pi}\} = 2^{\frac{1}{16}} e^{-\frac{\pi}{12}} (\sqrt{2}-1)^{-\frac{1}{4}} \quad \ldots \quad (3)$$

$$\therefore \quad Q_3' = \prod_1^\infty \{1 - e^{-2(2n-1)\pi}\} = 2^{\frac{3}{8}} e^{-\frac{\pi}{12}} \quad \ldots \quad \ldots \quad (4)$$

and
$$Q_1' = \prod_1^\infty \{1 + e^{-4n\pi}\} = 2^{-\frac{7}{16}} e^{\frac{\pi}{6}} (\sqrt{2}-1)^{\frac{1}{4}} \quad \ldots \quad (5)$$

In the second case, using the modular equation of the third order as in (*b*), *viz.*,

$$(\sin\alpha \sin\beta)^{\frac{1}{4}} + (\cos\alpha \cos\beta)^{\frac{1}{4}} = 1,$$

we have
$$\sqrt[8]{2} = (\sin\beta)^{\frac{1}{4}} + (\cos\beta)^{\frac{1}{4}}.$$

Also,
$$1 = \sin^2\beta + \cos^2\beta.$$

Solving,
$$(\sin\beta)^{\frac{1}{4}} = 2^{-\frac{3}{4}} + 2^{-\frac{5}{4}} 3^{\frac{1}{4}} (\sqrt{3}-1)$$

and
$$(\cos\beta)^{\frac{1}{4}} = 2^{-\frac{3}{4}} - 2^{-\frac{5}{4}} 3^{\frac{1}{4}} (\sqrt{3}-1),$$

whence
$$\sin 2\beta = 2^{-2}(\sqrt{3}-1)^4.$$

$$\therefore \quad Q_2' = 2^{\frac{1}{4}} e^{-\frac{3\pi}{24}} (\sin 2\beta)^{-\frac{1}{12}}$$

$$= 2^{\frac{1}{4}} e^{-\frac{3\pi}{24}} . 2^{\frac{1}{6}} (\sqrt{3}-1)^{-\frac{1}{3}}.$$

$$\text{Thus } Q_2' = \prod_1^\infty (1 + e^{-3(2n-1)\pi}) = 2^{\frac{5}{12}} e^{-\frac{\pi}{8}} (\sqrt{3}-1)^{-\frac{1}{3}} \quad \ldots \quad (6)$$

$$\therefore \quad Q_3' = \prod_1^\infty (1 - e^{-3(2n-1)\pi}) = 2^{\frac{1}{24}} e^{-\frac{\pi}{8}} (\sqrt{3}-1)^{-\frac{1}{3}} \times$$

$$\{1 - 2^{\frac{1}{2}} 3^{\frac{1}{4}} (\sqrt{3}-1)\}^{\frac{1}{2}} \ldots \quad (7)$$

and $$Q_1' = \prod_1^\infty (1 - e^{-6n\pi}) = 2^{-\frac{11}{24}} e^{\frac{\pi}{4}} (\sqrt{3} - 1)^{\frac{2}{3}} \times \left\{ 1 - 2^{-\frac{1}{3}} 3^{\frac{1}{4}} (\sqrt{3} - 1) \right\}^{-\frac{1}{2}} \quad \ldots \quad (8)$$

A similar procedure may be applied to the other results given at the end of Part I of the paper cited at the beginning.

§ 3. We shall now proceed to show how the equations (*a*), (*b*) and (*c*) could be applied to evaluate separately the two parts Σ_1 and Σ_2 which appeared together in the results VII to XII of Part II of the aforesaid paper. Result VII is

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{13}{3}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{13}{3}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{13}{3}}} - \ldots\right]$$

$$- \left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{13.3}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{13.3}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{13.3}} - \ldots\right]$$

$$= \frac{\pi}{12} - \tfrac{1}{2} \tan^{-1} \sqrt{\frac{3}{13}}. \quad \ldots \quad (f)$$

Now, it is known that

$$\sum_1^\infty (-1)^{n+1} \tan^{-1} \left(e^{-(2n-1)\frac{\pi}{2} \cdot \frac{k'}{k}}\right) = \tfrac{1}{4} \sin^{-1} (k)^*$$

where $k = \sin \alpha$, α being the modular angle.

Hence, if we suppose the first part of the series on the L. H. S., *viz.*,

$$\sum_1^\infty (-1)^{n+1} \tan^{-1} \left(e^{-(2n-1)\frac{\pi}{2} \cdot \sqrt{\frac{13}{3}}}\right) = \frac{\alpha}{4},$$

then the 2nd part, *viz.*,

$$\sum_1^\infty (-1)^{n+1} \tan^{-1} \left(e^{-(2n-1)\frac{\pi}{2} \cdot \sqrt{13.3}}\right) = \frac{\beta}{4},$$

where α and β are connected by the modular equation of the 3rd order, *viz.*,

$$(\sin \alpha \sin \beta)^{\frac{1}{2}} + (\cos \alpha \cos \beta)^{\frac{1}{2}} = 1.$$

This could be written as

$$(\sin \alpha \sin \beta)^{\frac{1}{2}} = 1 - (\cos \alpha \cos \beta)^{\frac{1}{2}}.$$

* *Vide* Jacobi: *Fundamenta Nova.*

Squaring and transposing,

$$2(\cos\alpha\cos\beta)^{\frac{1}{2}} = 1 + \cos(\alpha+\beta).$$

Squaring again,

$$4\cos\alpha\cos\beta = 1 + 2\cos(\alpha+\beta) + \cos^2(\alpha+\beta).$$

$$\therefore\ 2\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\} = 1+2\cos(\alpha+\beta)+\cos^2(\alpha+\beta);$$

i.e.,
$$2\cos(\alpha-\beta) - 2 = \cos^2(\alpha+\beta) - 1$$

i.e.,
$$-4\sin^2\frac{\alpha-\beta}{2} = -\sin^2(\alpha+\beta)$$

i.e.,
$$\sin(\alpha+\beta) = 2\sin\frac{\alpha-\beta}{2}\dagger \quad \ldots \quad \ldots \quad \text{(D)}$$

convenient form used below.

We have, from (f),

$$\frac{\alpha}{4} - \frac{\beta}{4} = \frac{\pi}{12} - \tfrac{1}{2}\tan^{-1}\sqrt{\frac{3}{13}}$$

$$= \frac{\pi}{12} - \tfrac{1}{2}\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$$

$$\therefore\ \frac{\alpha-\beta}{2} = \frac{\pi}{6} - \sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$$

$$= \sin^{-1}\left(\frac{\sqrt{13}-3}{8}\right).$$

But, from (D),

$$\alpha+\beta = \sin^{-1}\left\{2\sin\frac{\alpha-\beta}{2}\right\}$$

$$= \sin^{-1}\left(\frac{\sqrt{13}-3}{4}\right).$$

$$\therefore\ \frac{\alpha+\beta}{2} = \tfrac{1}{2}\sin^{-1}\left(\frac{\sqrt{13}-3}{4}\right).$$

Using this with

$$\frac{\alpha-\beta}{2} = \sin^{-1}\left(\frac{\sqrt{13}-3}{8}\right),$$

† This form is mentioned by the late S. Ramanujan, F. R. S., *vide J. I. M. S.* Vol. IV, p. 78.

we have,

$$\alpha = \tfrac{1}{2} \sin^{-1} \left(\frac{\sqrt{13} - 3}{4}\right) + \sin^{-1} \left(\frac{\sqrt{13} - 3}{8}\right)$$

and

$$\beta = \tfrac{1}{2} \sin^{-1} \left(\frac{\sqrt{13} - 3}{4}\right) - \sin^{-1} \left(\frac{\sqrt{13} - 3}{8}\right).$$

Hence,

$$\sum_1^\infty (-1)^{n+1} \tan^{-1} \left(e^{-(2n-1)\frac{\pi}{2}\cdot\sqrt{\frac{13}{3}}}\right).$$

$$= \tfrac{1}{8} \sin^{-1} \left(\frac{\sqrt{13} - 3}{4}\right) + \tfrac{1}{4} \sin^{-1} \left(\frac{\sqrt{13} - 3}{8}\right). \quad (9)$$

This is a solution of the second part of Question 684 in the *J. I. M. S.**

Also, $$\sum_1^\infty (-1)^{n+1} \tan^{-1} \left(e^{(2n-1)\frac{\pi}{2}\cdot\sqrt{13.3}}\right)$$

$$= \tfrac{1}{8} \sin^{-1} \left(\frac{\sqrt{13} - 3}{4}\right) - \tfrac{1}{4} \sin^{-1} \left(\frac{\sqrt{13} - 3}{8}\right). \quad (10)$$

Again, Result XXV of the paper cited above gives

$$\sum_1^\infty (-1)^{n+1} \tan^{-1} \left(e^{-(2n-1)\frac{\pi}{2}\cdot\frac{k'}{k}}\right) + \sum_1^\infty (-1)^{n+1} \tan^{-1} \left(e^{-(2n-1)\frac{\pi}{2}\cdot\frac{k}{k'}}\right)$$

$$= \frac{\pi}{8}.$$

$$\therefore \quad \sum_1^\infty (-1)^{n+1} \tan^{-1} \left(e^{-(2n-1)\frac{\pi}{2}\cdot\sqrt{\frac{3}{13}}}\right)$$

$$= \frac{\pi}{8} - \tfrac{1}{8} \sin^{-1} \left(\frac{\sqrt{13} - 3}{4}\right) - \tfrac{1}{4} \sin^{-1} \left(\frac{\sqrt{13} - 3}{8}\right). \quad \ldots (11)$$

To results IX and X of the same paper, we could similarly apply the modular equation of the 3rd order, by utilising result VI. But this work has already been done by Mr. M. Bhimasena Rao in his paper on "The Modular Equation of the Third Order."†

Next, taking result XI of the joint paper,

$$\left\{ \tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{5}{11}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{5}{11}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{5}{11}}} - \ldots\ldots \right\}$$

$$- \left\{ \tan^{-1} e^{-\frac{\pi}{2}\sqrt{5.11}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{5.11}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{5.11}} - \ldots \right\}$$

$$= \frac{\pi}{4} - \tfrac{1}{2} \tan^{-1} \sqrt{\frac{11}{5}}.$$

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* *Vide*: *J. I M. S.*, Vol. VII, No. 4.

† *Vide*: *J. I. M. S.*, Vol. XV, No. 6, results on p. 135,

Now, let

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{11}{5}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{11}{5}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{11}{5}}} - \ldots = \tfrac{1}{4}\alpha. \qquad \text{(G)}$$

Then, from result XXV of the joint paper

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{5}{11}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{5}{11}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{5}{11}}} - \ldots = \frac{\pi}{8} - \frac{1}{4}\alpha; \qquad \text{(H)}$$

and also

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{5.11}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{5.11}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{5.11}} - \ldots = \frac{1}{4}\beta,$$

where β and α are connected by the modular equation of the fifth order of which we may use the form (2), *viz.*,

$$\sin(\alpha + \beta) = 2\sin\frac{\alpha - \beta}{2} - \sin^3\frac{\alpha - \beta}{2}.$$

If $\dfrac{\alpha - \beta}{2} = \sin^{-1} p$, then $\alpha + \beta = \sin^{-1}(2p - p^3)$.

But, we have $\quad \dfrac{\pi}{8} - \dfrac{1}{4}\alpha - \dfrac{1}{4}\beta = \dfrac{\pi}{4} - \dfrac{1}{2}\tan^{-1}\sqrt{\dfrac{11}{5}}.$

$$\therefore \quad \alpha + \beta = \sin^{-1}\left(\tfrac{3}{8}\right).$$

$$\therefore \quad 2p - p^3 = \tfrac{3}{8}.$$

This gives an admissible solution $p = \dfrac{3 - \sqrt{5}}{4}$.

$$\therefore \quad \alpha + \beta = \sin^{-1}\tfrac{3}{8};\ \alpha - \beta = 2\sin^{-1}\frac{3 - \sqrt{5}}{4}$$

or

$$\frac{\alpha}{2} + \frac{\beta}{2} = \sin^{-1}\frac{\sqrt{11} - \sqrt{5}}{4\sqrt{2}},$$

and

$$\frac{\alpha}{2} - \frac{\beta}{2} = \sin^{-1}\frac{3 - \sqrt{5}}{4};$$

such that

$$\alpha = \sin^{-1}\frac{\sqrt{11} - \sqrt{5}}{4\sqrt{2}} + \sin^{-1}\frac{3 - \sqrt{5}}{4},$$

and

$$\beta = \sin^{-1}\frac{\sqrt{11} - \sqrt{5}}{4\sqrt{2}} - \sin^{-1}\frac{3 - \sqrt{5}}{4}.$$

Hence,

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{11}{5}}} - \tan^{-\frac{3\pi}{2}\sqrt{\frac{11}{5}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{11}{5}}} - \ldots\ldots$$

$$= \frac{1}{4}\left\{ \sin^{-1}\frac{\sqrt{11}-\sqrt{5}}{4\sqrt{2}} + \sin^{-1}\frac{3-\sqrt{5}}{4} \right\}, \quad \ldots \quad (12)$$

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{5.11}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{5.11}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{5.11}} - \ldots\ldots$$

$$= \frac{1}{4}\left\{ \sin^{-1}\frac{\sqrt{11}-\sqrt{5}}{4\sqrt{2}} - \sin^{-1}\frac{3-\sqrt{5}}{4} \right\}, \quad \ldots \quad (13)$$

and

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{5}{11}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{5}{11}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{5}{11}}} - \ldots\ldots$$

$$= \frac{1}{4}\left\{ \frac{\pi}{2} - \sin^{-1}\frac{\sqrt{11}-\sqrt{5}}{4\sqrt{2}} - \sin^{-1}\frac{3-\sqrt{5}}{4} \right\}. \quad \ldots \quad (14)$$

In conclusion, I desire to acknowledge my indebtedness to Mr. M. Bhimasena Rao at whose suggestion this paper was written.

On the relation between the Convergence of a Series and its Summability by Cesàro's Means*

By K. ANANDA RAU.

1. In the first of a series of important papers communicated to the London Mathematical Society, Hardy proved the following:

THEOREM I.† *If* Σa_n *is summable* (C 1) *to sum* s, *and if*

(1) $$a_n = O\left(\frac{1}{n}\right),$$

then Σa_n *converges to sum* s.

In case when the a's are real, Landau‡ made an extension of this theorem by replacing condition (1) by

$$a_n > -\frac{K}{n},$$

where K is a positive constant independent of n. The object of this paper is to make a further extension of the theorem in a somewhat different direction. The generalization consists in this, that we replace the sequence $\left(\frac{K}{n}\right)$ by a general sequence (b_n)§ and postulate on the asymptotic behaviour of the sum

(2) $$b_1 + b_2 + \dots\dots + b_n.$$

The proof of the generalized theorem is obtained by modifying that of Theorem I and its "left-handed" extension. It may make the genesis of the generalization clear, if I point out the relation between the restriction that is to be imposed on the sum (2) and the known properties of the sequence $\left(\frac{K}{n}\right)$.

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* A short abstract of this paper was printed in the Records of the Proceedings of the London Mathematical Society, Vol. 18 (1920), p. xxii.

† "Theorems relating to the Summability and Convergence of Slowly Oscillating Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301—320.

‡ *Prac Matematyczno-Fizycznych.* Vol 21 (1910), pp. 97—177.

§ The terms of this sequence are not necessarily positive.

We know that

$$\frac{K}{1} + \frac{K}{2} + \dots + \frac{K}{n} = K \log n + K' + o(1)$$

where $\psi(n) \equiv K \log n + K'$

satisfies the condition

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} [\psi(n + n\delta) - \psi(n)] = 0.$$

The restriction that we impose on the sum (2) is that, as $n \to \infty$,

$$\tau_n = \sum_1^n b_n = \chi(n) + o(1)$$

where $\chi(n)$ is a positive increasing function such that

(3) $$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} [\chi(n + n\delta) - \chi(n)] = 0$$

2. The theorem runs as follows :

THEOREM II. *Suppose that the sequence (b_n) satisfies the condition stated above. Let Σa_n be summable $(C\,1)$ to sum s, and let*

$$a_n + b_n \geqslant 0$$

for sufficiently large values of n. Then Σa_n converges to sum s.

Let $$s_n = a_1 + a_2 + \dots\dots + a_n$$

$$\sigma_n = \frac{s_1 + s_2 + \dots\dots + s_n}{n}.$$

We may clearly suppose that $s = 0$. If the theorem were false, there is an $h > 0$, such that one at least of the inequalities

(i) $s_n > h$

(ii) $s_n < -h$

is satisfied for an infinity of values of n. We shall prove that either of these suppositions leads to a contradiction. Let us consider, for instance, the supposition (i).

First choose a positive $\delta < 1$ so that

$$\phi(\delta) = \lim_{n \to \infty} [\chi(n + n\delta) - \chi(n)] < \frac{h}{48}.$$

Keeping δ fixed, choose N_1 so that for $n > N_1$

$$\chi(n + n\delta) - \chi(n) < \phi(\delta) + \frac{h'}{48};$$

we shall then have for $n > N_1$

$$\chi(n + n\delta) - \chi(n) < \frac{h}{24}. \tag{4}$$

Next choose N_2 so that for $n > N_2$

$$|\sigma_n| < \frac{1}{8}\,\frac{h\delta}{1+\delta}, \tag{5}$$

$$|\tau_n - \chi(n)| < \frac{h}{24}, \tag{6}$$

$$a_n + b_n \geqslant 0.$$

Let n be such that

$$n > \max(N_1, N_2).$$

$$s_n > h.$$

Let $r = r(n)$ be the greatest integer in $n\delta$.

Now, for $p = 1, 2, \ldots\ldots r$,

$$s_{n+p} = s_n + a_{n+1} + \ldots\ldots + a_{n+p}$$

$$> h - (b_{n+1} + \ldots\ldots + b_{n+p})$$

$$= h - (\tau_{n+p} - \tau_n). \tag{7}$$

$$\sigma_{n+r} = \frac{s_1 + s_2 + \ldots\ldots + s_n + \ldots\ldots + s_{n+r}}{n+r}$$

$$= \frac{s_1 + s_2 + \ldots\ldots + s_n}{n}\,\frac{n}{n+r} + \frac{s_{n+1} + \ldots\ldots + s_{n+r}}{n+r}$$

Hence by (5) and (7),

$$\sigma_{n+r} > -\frac{1}{8}\,\frac{h\delta}{1+\delta} + \frac{hr}{n+r} - \frac{1}{n+r}\sum_{p=1}^{r}(\tau_{n+p} - \tau_n) \tag{8}$$

Now

$$\frac{hr}{n+r} \geqslant \frac{1}{2}\,\frac{h(r+1)}{n+r} > \frac{h}{2}\,\frac{\delta}{1+\delta}, \tag{9}$$

since $\qquad r + 1 > n\delta,\ r \leqslant n\delta.$

Further,

(10) $$\left| \sum_{p=1}^{r} (\tau_{n+p} - \tau_n) \right| \leqslant \sum_{p=1}^{r} \left| \tau_{n+p} - \chi(n+p) \right| +$$

$$\sum_{p=1}^{r} \left| \tau_n - \chi(n) \right| + \sum_{p=1}^{r} \left| \chi(n+p) - \chi(n) \right|.$$

The first two sums on the right are each less than $\frac{hr}{24}$ by (6). As for the last term, we have

$$\sum_{p=1}^{r} \left| \chi(n+p) - \chi(n) \right| \leqslant r\, [\chi(n+r) - \chi(n)]$$

$$\leqslant r\, [\chi(n + n\delta) - \chi(n)]$$

$$< \frac{hr}{24}$$

by (4). Hence

(11) $$\frac{1}{n+r} \left| \sum_{p=1}^{r} (\tau_{n+p} - \tau_n) \right| < \frac{hr}{8(n+r)} < \tfrac{1}{8}\, \frac{h\delta}{1+\delta}.$$

Using (9), (10) and (11) in (8), we get

$$\sigma_{n+r} > -\tfrac{1}{8}\frac{h\delta}{1+\delta} + \tfrac{1}{2}\frac{h\delta}{1+\delta} - \tfrac{1}{8}\frac{h\delta}{1+\delta}$$

$$> \tfrac{1}{4}\frac{h\delta}{1+\delta}.$$

Clearly $(n+r) \rightarrow \infty$ as $n \rightarrow \infty$. Hence for an infinity of values of m, we have

$$\sigma_m > \frac{h\delta}{4(1+\delta)},$$

which is contradictory to the hypothesis that $\sigma_m \rightarrow 0$.

We may similarly dispose of the supposition (ii). In this case, we consider σ_{n-r} instead of σ_{n+r} and remembering that $\delta < 1$, $(n-r) \rightarrow \infty$, we prove that for an infinity of values of m

$$\sigma_m < -\frac{h\delta}{4(1+\delta)},$$

which is also contradictory to the hypothesis of the theorem.

The proof is therefore complete.

3. If for a given series Σa_n, we take

$$b_n = | a_n |$$

then the condition $$a_n + b_n \geqslant 0$$

is clearly satisfied. Also, if we write

$$\chi(n) = \sum_1^n | a_n |,$$

then $\chi(n)$ is a positive increasing function and the condition (3) becomes

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} \sum_n^{n(1+\delta)} | a_n | = 0.$$

Theorem II therefore reduces to the following :

THEOREM III. *Suppose that Σa_n is summable $(C\,1)$ and that*

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} \sum_n^{n(1+\delta)} | a_n | = 0;$$

then Σa_n is convergent.

The Group Concept

By G. A. Miller.

1. The word *group* seems to have originated in Central Europe and to be etymologically akin to the word 'crop.' A mathematical group may profitably be regarded as a mathematical crop for which seeds were being scattered very early in the development of our subject but which received little attention and was not thoughtfully cultivated until recent times. Under such cultivation it has yielded rich harvests whose importance has been increasingly recognized, especially during the last fifty years, and there seems to be no evidence which tends to show that we have as yet fully realised its value and its varied usefulness. New and important applications are being discovered in rapid succession and only few mathematicians are as yet in a position to evaluate their full significance.

The word group was in use for a long time before E. Galois began to use it as a technical mathematical term in about 1830. He used it also as a non-technical term in the sense of a collection, and later mathematical writers have followed him in this respect. For instance, we still speak of a group of points, a group of waves, group insurance, etc., as well as of a group of substitutions, a group of movements, a group of operations, etc. What is of still greater importance to the beginner in this field of study is the fact that, even in the best modern mathematical literature, the technical term group is being used with widely different meanings as may be seen from the different definitions of this term in the *Encyclopédie des Sciences Mathematiques*, tome 1, volume 1, page 576; and tome 1, volume 2, page 243.

2. The most general technical mathematical meaning of the word group may be obtained by restricting its non-technical meaning: that is, by imposing the condition that each of the elements of the collection can be combined with itself and also with every other element of the collection. Moreover, the result obtained by such a combination must always be an element of the collection. This closure property is involved in every

mathematical definition of the term group, and it was the only one imposed on the elements under consideration by the early writers. This was due to the fact that these elements were concrete objects which by their nature satisfy the other conditions which are now usually imposed on a set of abstract elements which constitute a group.

It should be noted that the concept of group appeared in the mathematical literature in an implicit form more than three thousand years before any special name was assigned to this concept in its modern abstract form. In a special concrete form the concept has had various names. The earliest of these names was doubtless the name for unity. In fact, the positive rational numbers, exclusive of zero, constitute a group in the modern restricted sense of the term with respect to the operation known as multiplication, and all the number systems of the various civilizations existing before the Christian era were included in this group of numbers, which may be regarded as the most important mathematical group. From the work of Ahmes, written about 1700 B.C., it is clear that the ancient Egyptians had acquired considerable insight into this group and some of its sub-groups. Their word for number may possibly have represented this group.

The sub-group of the positive rational number group, which is formed by the powers of 10, played a very fundamental role in the development of the various number systems. Among the Babylonians and the ancient Greeks, the sub-group formed by the powers of 60 received especial attention, and this sub-group receives still some attention in our measurement of time and angles. It would doubtless be claiming too much to say that the ancient Egyptians and the ancient Babylonians had a clear notion of the group formed by the positive rational numbers. In fact, even the ancient Greeks failed to be sufficiently influenced by the properties of this group and its various sub-groups, otherwise they would have introduced negative numbers. Such numbers are essential not only for the study of the sub-groups—in the modern restricted sense of the term group—formed by certain arithmetical series, but also of those formed by certain geometrical series, including those formed by the powers of 10 and the powers of 60.

Groups have been divided into *four* categories, as follows: finite discontinuous, infinite discontinuous, finite continuous and infinite continuous. All the groups and sub-groups noted above, with the exception of the group formed by unity, belong to the second of these categories. Notwithstanding the fact that, with the exception of trivial cases, the earliest groups which played an important part in the development of mathematics were of infinite order yet it was the finite discontinuous groups which first led to a general theory of groups. These groups began to be studied during the latter half of the eighteenth century in connection with the solution of algebraic equations in one unknown, and the terminology thus developed was largely transferred to similar concepts arising in connection with the study of the other categories. Among the important concepts common to all of these categories are the following: sub-group, invariant sub-group, conjugate groups and quotient groups.

3. We referred above to the most general technical definition of the term group. The most common additional restriction imposed on the elements of a group is that each element of the collection can be combined with at least one element of the collection so as to obtain the identity. In other words, each of the elements of the group shall have at least one inverse. It should be noted that in Lie's theory of transformation groups this restriction is not always imposed. In what follows we shall assume that when the elements of a group are combined the associative law is satisfied, and if any two of the symbols of the equation

$$xy = z$$

are replaced by elements of the group the equation has always one and only one root in the group, which is assumed to be composed of distinct elements. This equational definition of the term group appears to the writer to be the most satisfactory of all those which have been advanced.

4. A central feature of the development of group theory has been the establishment of an $(\infty, 1)$ correspondence. While there is an infinite number of algebraic equations of degree n there is only a finite number of groups on n letters, and many of the fundamental properties of these equations are reflected in the properties of the corresponding groups.

The theory of the possible substitution groups on n letters may appear to involve many details; but when it is noted that each of these possible groups reflects light on an infinite system of equations these details assume a greater significance. In particular, during recent years a theory relating to the cycles of the substitutions involved in a group has received some attention. It is obvious that such a theory can only be expected to attract general interest on account of the $(\infty, 1)$ correpondence between substitution groups and various other fields of mathematical interest.

The number of elements involved in a finite discontinuous group is said to be its *order* and the existence of at least one group of any given finite order n is established by the fact that the n rotations through an angle $2\pi/n$ and its multiples obviously constitute a group of order n. We thus have a very simple existence theorem relating to the group concept, and the establishment of existence theorems as regards other possible groups has constituted a very fruitful field of investigation. While these investigations have led to many new and unsuspected relationships they have not encouraged those who would like to think that the intellectual space of human endeavour is finite. On the contrary, new difficulties present themselves much more rapidly here than the overcoming of old ones. The growing horizon of intellectual penetration becomes also a growing horizon of intellectual difficulties. In particular, there is as yet no basis in sight for the hope that we shall ever know all the simple discontinuous groups of finite order.

5. While existence theorems as regards possible groups have played a fundamental role in the development of the group concepts, a still more fundamental role has been played by the wide applications of this concept to various branches of mathematics. It is probable that H. Poincare had such applications in mind when he said towards the close of his life: "The theory of groups is, so to say, entire mathematics divested of its matter and reduced to pure form."* Such applications seem also to have been in the mind of E. Study when he said, :" The word elegance denotes only to some extent what is involved here. It is

* *Acta Mathematica*, Vol. 38 (1921), p. 145.

a colossal misunderstanding to regard as only outside matters that which is at stake as regards the innermost character of mathematics. It is a question of conformity to laws which are mostly of a group theoretic character." *

In a recent number of *Scientia,* Volume 18 (1924), page 145, G. Fano published an article entitled: "*I gruppi di transformazione nella geometria*" in which he points out the fundamental position of the group concept in the development of geometry. In particular, he again calls attention to the fact that the best answer to the question 'what is a geometric property?' is one which starts from the concept of group of transformations (or of operations), a fundamental concept which during a century is on the way of becoming very important both as a source of information and as a principle for classifying the ancient and recent theories. Later in the same article, he asserts that the concept of group of operations is one of those which govern all modern mathematics. The present writer has often been surprised to find so many other very emphatic statements relating to the importance of the concept of groups, and hence he published a brief list of such statements in the *Tohoku Mathematical Journal,* Vol. 6 (1914), page 73; a somewhat more extended list was later published in *Revista Matematica Hispano-Americano,* Volume 1 (1919), page 148. It would probably be impossible to provide an equally imposing list of favourable comments on the part of leading mathematicians relating to any other modern mathematical concept.

6. An essential element of the concept of group is that it deals with a totality of operations. The simplest such totality consists of the identity operation and the group constituted by it is sometimes called the *principal group* (*Hauptgruppe*). F. Klein used the same term for the continuous mixed group of movements, similitudes and symmetries—the study of whose invariants constitutes elementary geometry. In a certain sense the identity operation is no operation at all and it is not known who first regarded it as an operation. Since every group must involve the identity operation, it is obvious that *the concept of group with its modern restricted*

* E. Study: *Einleitung in die Theorie der Invarianten linearer Transformationen auf Grund der Vektorenrechnung* 1923, **Introduction.**

meaning could not have existed prior to the concept of the identity operation; and conversely. In the group formed by the positive rational numbers when they are combined by multiplication, the number 1 constitutes the identity operation and this fact must have been observed very early in the development of the number concept.

7. When we stated above that the oldest name of a group was the name for unity it was not implied that those who first used this name were impressed by the fact that the number 1 is a group as regards the operation of multiplication. It may be that the group property of the number 1 as regards the operation of multiplication was not observed any earlier than that of the positive rational numbers when they are combined according to the same operation. In the former case, this property appears at first to be more trivial than in the latter, and hence it is conceivable that these two groups were discovered at the same time, but the former doubtless received a name long before it was regarded as a group. As regards the latter, it is not so clear that the name number was assigned to it before its group property with respect to multiplication was observed. Mathematicians have probably always differed widely as regards keen distinctions. It is of interest to know when people first began to use groups even though it is impossible to determine whether they were aware of their existence, or whether they were even vaguely guided thereby.

8. The growing popularity, rather than the growing appreciation, of the group concept is reflected in the number of books explicitly devoted thereto. The first of these was C. Jordan's *Traite des substitutions et des equations algebraiques*, 1870. This was the first book published by its author (with the exception of his thesis), and represents a field in which he was interested until the end of his very fruitful mathematical career. It contains much that was new and it inspired much of the later work along the same line; but it was never republished in French or any other language. About 12 years after its publication, Netto's *Substitutionentheorie* (1882) appeared. This was a more elementary work. An Italian edition of it, prepared by G. Battaglini, appeared in 1885, and an English edition pre-

pared by F. N. Cole appeared in 1892. Both of these works were devoted to finite discontinuous groups; but the next two large works which were explicitly devoted to the theory of groups were devoted to finite continuous groups. The former of these appeared in three volumes, entitled *Theorie der Transformationsgruppen* (1888—1893), by S. Lie and F. Engel, while the latter appeared in a single volume under the title *Vorlesungen uber kontinuierliche Gruppen*, 1893, by S. Lie and G. Scheffers.

After the publication of these four works other books devoted explicitly to the group concept began to appear more rapidly; but the number of these books is still small, since the number of students who have made an extensive study of this concept has always been relatively small. Among the books which aim to treat a relatively large portion of the subject are the following:—*Theory of Groups of Finite Order*, 1897, by W. Burnside, second edition, 1911; *Nozioni sulle transformazioni puntuali e sui gruppi continui*, 1895, by F. Giudice; *Teoria dei gruppi di transformazioni*, 1898, by G. Vivanti, translated into French by A. Boulanger, 1904; *Lezioni sulla teoria dei gruppi di sostituzioni*, 1900, by L. Bianchi; *Linear groups*, 1901, by L. E. Dickson; *Theory of continuous groups*, 1903, by J. E. Campbell; *Groupes abstraits*, 1904, by J. A. de Seguier; *Teoria dei gruppi discontinui e delle funzioni automorfe*, 1908, by G. Fubini; *Finite groups*, 1908, by H. Hilton; *Gruppen-und substitutionentheorie*, 1908, by E. Netto: *Groupes de substitutions*, 1912, by J. A, de Seguier; *Theory and applications of finite groups*, 1916, by Miller, Blichfeldt and Dickson; *Lezioni sulla teoria dei gruppi continui finiti di transformazioni*, 1918, by L. Bianchi; *Theorie der Gruppen von endlicher Ordnung*, 1923, by A. Speiser.

195340

THE JOURNAL

OF

THE INDIAN MATHEMATICAL SOCIETY

Volume XV, No. 12.

December 1924.

NOTES AND QUESTIONS.

Notes and Questions.

The Six Co-ordinates of a Right Line.

Introduction. The equations to a line passing through the point (a, b, c) and having for its direction cosines (l, m, n) are

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}; \qquad \ldots \text{ (i)}$$

whence it follows that

$$\left.\begin{aligned} ny - mz &= nb - mc = \lambda, \\ lz - nx &= lc - na = \mu, \\ \text{and} \quad mx - ly &= ma - lb = \nu. \end{aligned}\right\} \qquad \ldots \text{ (ii)}$$

The quantities λ, μ, ν are constant for any point on the line. Hence the six constants of a line are $(l, m, n, \lambda, \mu, \nu)$. These are termed "line co-ordinates." Also equations (i) can be written in the form

$$\left.\begin{aligned} x &= Az + B \\ \text{and} \quad y &= Cz + D \end{aligned}\right\},$$

from which it is evident that the equations to a line involve *four* and only four *independent* constants.

Hence we must have two identical relations among the six constants of a line. These are

$$l^2 + m^2 + n^2 = 1 \text{ and } l\lambda + \mu m + n\nu = 0.$$

It may be noted that if p denotes the perpendicular from the origin on the line and (L, M, N) are the direction cosines of the normal to the plane passing through the origin and the line, then

$$\lambda = pL, \ \mu = pM, \ \nu = pN.$$

If the line passes through the origin, it is obvious from (ii), that $\lambda = \mu = \nu = 0$. Take (x, y, z) any point not necessarily on the line and let

$$\begin{aligned} A &\equiv ny - mz - \lambda, \\ B &\equiv lz - nx - \mu, \\ \text{and} \quad C &\equiv mx - ly - \nu. \end{aligned}$$

Then for any point on the line

$$A = 0, B = 0 \text{ and } C = 0.$$

Hence A = 0, B = 0 and C = 0 represent planes through the line parallel to OX, OY and OZ respectively.

In fact, the line $(l, m, n, \lambda, \mu, \nu)$ may be defined as the intersection of any two of the planes A = 0. B = 0, C = 0.

Let $$D \equiv \lambda x + \mu y + \nu z.$$

Then $$Ax + By + Cz + D \equiv 0$$

for any point on the line A = B = C = 0.

Hence D = 0, or $\lambda x + \mu y + \nu z = 0$ is the equation to the plane passing through the line and the origin.

Take any fixed point (x_1, y_1, z_1) and write

$$A_1 \text{ for } ny_1 - mz_1 - \lambda,$$
$$B_1 \text{ for } lz_1 - nx_1 - \mu,$$
and $$C_1 \text{ for } mx_1 - ly_1 - \nu.$$

Also, let $$D_1 \equiv \lambda x_1 + \mu y_1 + \nu z_1.$$

Now consider $A_1x + B_1y + C_1z + D_1 = 0$.
This represents a plane through $(x_1\, y_1\, z_1)$; for

$$A_1 x_1 + B_1 y_1 + C_1 z_1 + D_1 \equiv 0.$$

Also $A_1x + B_1y + C_1z + D_1 \equiv x_1A + y_1B + z_1C + D.$

Further $$Ax_1 + By_1 + Cz_1 + D = 0$$

is satisfied by any point on the line.

Hence the equation $A_1 x + B_1 y + C_1 z + D_1 = 0$ represents a plane through the point $(x_1\, y_1\, z_1)$ and the line $(l, m, n, \lambda, \mu, \nu)$.

§ 1. *To find the condition that two given lines* $(l, m, n, \lambda, \mu, \nu)$ *and* $(l', m', n', \lambda', \mu', \nu')$ *are coplanar.*

If possible let them meet in (x_1, y_1, z_1). Then

$$\left.\begin{aligned} ny_1 - mz_1 - \lambda &= 0, \\ lz_1 - nx_1 - \mu &= 0, \\ mx_1 - ly_1 - \nu &= 0; \end{aligned}\right\} \text{ and } \left.\begin{aligned} n'y_1 - m'z_1 - \lambda' &= 0, \\ l'z_1 - n'x_1 - \mu' &= 0, \\ m'x_1 - l'y_1 - \nu' &= 0. \end{aligned}\right\}.$$

Out of these six equations if $x_1\, y_1\, z_1$ be eliminated, three conditions among the constants must result, out of which two are

$$l\lambda + m\mu + n\nu = 0, \text{ and } l'\lambda' + \mu'm' + n'\nu' = 0. \quad \ldots \quad (1)$$

Hence one more condition is necessary for the intersection of the two lines.

Multiply the first set of equations by l', m', n' in order and the second set by l, m, n in order and add; then we get

$$l'\lambda + m'\mu + n'\nu + l\lambda' + m\mu' + n\nu' = 0. \quad \dots \quad (2)$$

Supposing this condition is satisfied it is easy to write out the equation to the plane containing the given lines. If the direction cosines of the normal to this plane are a, b, c, then

$$al + bm + cn = 0,$$
$$al' + bm' + cn' = 0.$$

$$\therefore \quad \frac{a}{mn' - m'n} = \frac{b}{nl' - n'l} = \frac{c}{lm' - l'm}.$$

Also to determine the plane completely, we must know a fixed point through which it passes, and the direction cosines of its normal.

This point can be chosen on any one of the given lines, say the point where the first line meets the plane $x = 0$, *viz.*, $\left(0, \frac{-\nu}{l}, \frac{\mu}{l}\right)$.

The equation of the plane is

$$\begin{vmatrix} x & y + \frac{\nu}{l}, & z - \frac{\mu}{l} \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0.$$

That is,

$$x(mn' - m'n) + y(nl' - n'l) + z(lm' - l'm)$$
$$+ \left(\frac{\nu l'n}{l} - \nu n' - \mu m' + \frac{\mu m l'}{l}\right) = 0.$$

The constant term, by the help of (2), becomes

$$\frac{\nu l'n}{l} + \frac{m\mu l'}{l} + l'\lambda + l\lambda' + \mu' m + n\nu'$$
$$= l\lambda' + m\mu' + n\nu', \qquad \text{by (1)}$$
$$= -(l'\lambda + m'\mu + n'\nu).$$

Hence the equation to the plane becomes

$$x(mn' - m'n) + y(nl' - n'l) + z(lm' - l'm)$$
$$+ (l\lambda' + m\mu' + n\nu') = 0,$$

or

$$x(mn' - m'n) + y(nl' - n'l) + z(lm' - l'm)$$
$$- (l'\lambda + m'\mu + n'\nu) = 0.$$

§ 2. *To find the perpendicular distance from $(x_1\, y_1\, z_1)$ to the line $(l, m, n, \lambda, \mu, \nu)$.*

Let the perpendicular meet the line in $(x_2\, y_2\, z_2)$. Then

$$\left.\begin{aligned} ny_2 - mz_2 - \lambda &= 0 \\ lz_2 - nx_2 - \mu &= 0 \\ mx_2 - ly_2 - \nu &= 0. \end{aligned}\right\}$$

Also let

$$\left.\begin{aligned} A_1 &\equiv ny_1 - mz_1 - \lambda \\ B_1 &\equiv lz_1 - nx_1 - \mu \\ C_1 &\equiv mx_1 - ly_1 - \nu. \end{aligned}\right\}$$

Then

$$\begin{aligned} A_1 &= (ny_1 - mz_1 - \lambda) - (ny_2 - mz_2 - \lambda) \\ &= n\,(y_1 - y_2) - m\,(z_1 - z_2). \end{aligned}$$

Similarly

$$\begin{aligned} B_1 &= l\,(z_1 - z_2) - n\,(x_1 - x_2), \\ C_1 &= m\,(x_1 - x_2) - l\,(y_1 - y_2). \end{aligned}$$

Now square and add, then

$$A_1^2 + B_1^2 + C_1^2 = \Sigma\,(x_1 - x_2)^2.$$

But $$p^2 = \Sigma\,(x_1 - x_2)^2.$$

$\therefore$ $$\begin{aligned} p^2 &= A_1^2 + B_1^2 + C_1^2 \\ &= \Sigma\,(ny_1 - mz_1 - \lambda)^2. \end{aligned}$$

If we replace $(x_1\, y_1\, z_1)$ by current co-ordinates, we find the equation of a right circular cylinder of radius p, of which the given line is the axis. Hence the equation to the cylinder will be

$$p^2 = \Sigma\,(ny - mz - \lambda)^2.$$

Corollary. To find the equation of a cone of semi-vertical angle α, having the given line as axis and a point O $(x_1\, y_1\, z_1)$ on it as its vertex: Take P $(x\, y\, z)$ any point on the cone, then the square of the length of the perpendicular from $(x\, y\, z)$ on the line is $\Sigma\,(ny - mz - \lambda)^2$, and this must be equal to $OP^2 \tan^2 \alpha$.

Hence $$[\Sigma\,(x - x_1)^2] \tan^2 \alpha = \Sigma\,(ny - mz - \lambda)^2,$$
is the equation of the cone required.

§ 3. *To find the shortest distance between two non-intersecting lines, $(l_1, m_1, n_1, \lambda_1, \mu_1, \nu_1)$ and $(l_2, m_2, n_2, \lambda_2, \mu_2, \nu_2)$.*

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the feet of the common perpendicular, and let (l, m, n) be its direction cosines.

Then $$l l_1 + m m_1 + n n_1 = 0,$$
and $$l l_2 + m m_2 + n n_2 = 0;$$
so that

$$\frac{l}{m_1 n_2 - m_2 n_1} = \frac{m}{n_1 l_2 - n_2 l_1} = \frac{n}{l_1 m_2 - l_2 m_1}$$
$$= \frac{1}{\sqrt{\Sigma (m_1 n_2 - m_2 n_1)^2}} = \frac{1}{\sin \theta}, \quad \ldots \quad (1)$$

where θ is the angle between the two straight lines. Also if p be the shortest distance between the two lines

$$\frac{x_1 - x_2}{p} = l, \quad \frac{y_1 - y_2}{p} = m, \quad \frac{z_1 - z_2}{p} = n. \quad \ldots \quad (2)$$

Again, $$\left.\begin{aligned} n_1 y_1 - m_1 z_1 - \lambda_1 &= 0 \\ l_1 z_1 - n_1 x_1 - \mu_1 &= 0 \\ m_1 x_1 - l_1 y_1 - \nu_1 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} n_2 y_2 - m_2 z_2 - \lambda_2 &= 0 \\ l_2 z_2 - n_2 x_2 - \mu_2 &= 0 \\ m_2 x_2 - l_2 y_2 - \nu_2 &= 0 \end{aligned}\right\} \quad (3)$$

Multiply the first set by l_2, m_2, n_2 in order and the second set by l_1, m_1, n_1 in order and add, then

$$\Sigma (m_1 n_2 - m_2 n_1) \ (x_1 - x_2) = l_1 \lambda_2 + m_1 \mu_2 + n_1 \nu_2 + l_2 \lambda_1 + m_2 \mu_1 + n_2 \nu_1;$$

or, using (1) and (2), we get

$$l \sin \theta \,.\, pl + m \sin \theta \,.\, p.m + n \sin \theta \,.\, p.n = l_1 \lambda_2 + m_1 \mu_2 + n_1 \nu_2 + l_2 \lambda_1 + m_2 \mu_1 + n_2 \nu_1;$$

$$\therefore \quad p (l^2 + m^2 + n^2) \sin \theta = l_1 \lambda_2 + m_1 \mu_2 + n_1 \nu_2 + l_2 \lambda_1 + m_2 \mu_1 + n_2 \nu_1.$$

$$\therefore \quad p = \frac{l_1 \lambda_2 + m_1 \mu_2 + n_1 \nu_2 + l_2 \lambda_1 + m_2 \mu_1 + n_2 \nu_1}{\sin \theta}.$$

Cor. 1. From this result we can at once determine the condition for the intersection of two lines. For, if the two lines intersect, $p = 0$,

$$\therefore \quad l_1 \lambda_2 + m_1 \mu_2 + n_1 \nu_2 + l_2 \lambda_1 + m_2 \mu_1 + n_2 \nu_1 = 0.$$

Cor. 2. If, however, the two lines are parallel to each other, they intersect each other at infinity and hence

$$l_1 \lambda_2 + m_1 \mu_2 + n_1 \nu_2 + l_2 \lambda_1 + m_2 \mu_1 + n_2 \nu_1 = 0.$$

Also θ, the angle between the lines, is zero. Hence the above method to determine the shortest distance between the lines fails; we investigate p for such lines in the next article.

§ 4. *To find the shortest distance between two parallel lines* $(l, m, n, \lambda_1, \mu_1, \nu_1), (l, m, n, \lambda_2, \mu_2, \nu_2)$.

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the feet of any line perpendicular to both, one on each.

Then $$p^2 = \Sigma (x_1 - x_2)^2.$$
Also $$ny_1 - mz_1 - \lambda_1 = 0,$$
and $$ny_2 - mz_2 - \lambda_2 = 0.$$
$$\therefore \quad n(y_1 - y_2) - m(z_1 - z_2) = \lambda_1 - \lambda_2.$$
Similarly $$l(z_1 - z_2) - n(x_1 - x_2) = \mu_1 - \mu_2.$$
and $$m(x_1 - x_2) - l(y_1 - y_2) = \nu_1 - \nu_2.$$
Also $$l(x_1 - x_2) + m(y_1 - y_2) + n(z_1 - z_2) = 0,$$

because the line joining $(x_1 y_1 z_1)$ and $(x_2 y_2 z_2)$ is perpendicular to each of the given lines.

Squaring and adding these four equations, we get
$$\Sigma (x_1 - x_2)^2 = (\lambda_1 - \lambda_2)^2 + (\mu_1 - \mu_2)^2 + (\nu_1 - \nu_2)^2$$
i.e., $$p^2 = \Sigma (\lambda_1 - \lambda_2)^2.$$

§ 5. *To find the condition that three straight lines may be concurrent.*

Of course the three conditions of intersection of the lines taken in pairs must first be satisfied. Then, supposing the lines meet at $(x' \; y' \; z')$, we have also

then
$$\lambda_1 x' + \mu_1 y' + \nu_1 z' = 0$$
$$\lambda_2 x' + \mu_2 y' + \nu_2 z' = 0$$
$$\lambda_3 x' + \mu_3 y' + \nu_3 z' = 0.$$

For the equation $\lambda x + \mu y + \nu z = 0$ is satisfied by the co-ordinates of any point on the line.

Eliminating $(x' \; y' \; z')$, we get the required condition in the form

$$\begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix} = 0.$$

P. L. SRIVASTAVA.

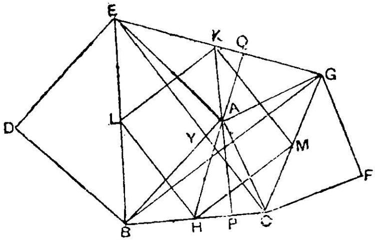
Solutions.

Question 1286.

(B. B. BAGI) :—Squares ABDE and ACFG are described externally on the sides AB, AC of a triangle ABC, L, M being their respective centres. H, K are the mid-points of BC, EG and AK, AH meet them in P, Q ; then show that (1) L, H, P, M, Q, K lie on a circle of which HK LM are perpendicular diameters and that (2) four times the area of square HMKL = sq. AF + sq. AD + 4 △ ABC.

Solution by I. B. Mukherjee, K. J. Sanjana, S. Audinarayanan, S. N. Kumaraswamy, V. A. Apte, K. R. Srinivasa Iyengar, and R. Srinivasan.

(1) The triangles EAC and BAG being congruent, EC and BG are equal and cut at right angles, for they are equally inclined in the same sense to EA and BA which form a right angle. L, H, M, K being the middle-points of the sides of EBCG form a parallelogram whose adjacent sides LH, HM are parallel and equal to $\frac{1}{2}$ EC and $\frac{1}{2}$ BG respectively The figure LHMK is therefore a square.



If GA is produced to N such that AN = GA, the triangles EAN and BAC are congruent; angle ACB = ENA = KAG (since EN is parallel to KA) = complement of PAC. Therefore APC is a right angle, as also AQG for a similar reason. Hence L, H, P, M, Q, K are concyclic having the diagonals of the square LM as perpendicular diameters.

(2) Again area of LHMK $= \frac{1}{2}$ EBCG.

$\therefore$ 4 (square LM) $= 2$ (ABE + ACG + ABC + AEG)
$=$ sq. AD + sq. AF + 4 ABC,

since ABC and AEG are equal in area, having two sides of the one equal to two sides of the other and the included angles supplementary.

Question 1295.

(M. BHIMASENA RAO) :— Solve,

$$(1 + x^2)^2 \frac{d^2y}{dx^2} + 4x(1 + x^2)\frac{dy}{dx} + (1 + 2x^2)\, y = 0.$$

Solution (1) by A. Mahalingam, K. N. Srikantha Sastry and M. V. Seshadri; (2) by Miss Y. Bhate, Hans R. Gupta, M. K. Kewalramani, I. B. Mukherjee, S. Audinarayanan, A. Mahadevan, J. B. Freeman, I. Totadri Iyengar and S. M. Shah.

(2) Changing the independent variable by the substitution $x = \tan\theta$, the equation reduces to

$$\frac{d^2y}{d\theta^2} + 2\tan\theta\frac{dy}{d\theta} + (1 + 2\tan^2\theta)\, y = 0.$$

This again, on putting $y = ve^{-\int \tan\theta\, d\theta}$, reduces to

$$\frac{d^2v}{d\theta^2} = 0.$$

$\therefore$ $y = (A + B\theta)\cos\theta$, A and B being constants of integration.

i.e.,
$$y = \frac{A + B\tan^{-1}x}{\sqrt{1 + x^2}}.$$

(2) The differential equation

$$\frac{d^2y}{dx^2} + \frac{4x}{1 + x^2}\frac{dy}{dx} + \frac{1 + 2x^2}{(1 + x^2)^2}\, y = 0$$

when reduced to its normal form becomes

$$\frac{d^2v}{dx^2} - \frac{1}{(1 + x^2)^2}\, v = 0.$$

Then following the particular method given in Forsyth's *Differential Equations*, § 68, p. 118, the differential equation

$$\frac{dP_1}{dx} - P_1^2 = -\frac{1}{(1 + x^2)^2}$$

has to be solved. Substitute

$$P_1 = \frac{a + bx}{1 + x^2}$$

for trial. If this be a solution the following three equations must be satisfied:

$$\text{(i)} \quad a^2 - b = 1,$$
$$\text{(ii)} \quad b(b + 1) = 0,$$
$$\text{(iii)} \quad 2a(b + 1) = 0;$$

$a = 0$, $b = -1$ satisfy these equations simultaneously.

$$\therefore \qquad P_1 = -\frac{x}{1 + x^2}.$$

Hence $$\frac{dz}{dx} + 2P_1 z = A,$$

where $$z = ve^{-\int P_1 dx} = v\sqrt{1 + x^2},$$

and A is an arbitrary constant.

This equation is linear in z. Integrating it,

$$z = (1 + x^2)\left\{ \int \frac{A dx}{1 + x^2} + B \right\}$$

$$= (1 + x^2)\left\{ A \tan^{-1} x + B \right\}$$

where B is another arbitrary constant.

Now $$z = v\sqrt{1 + x^2} = ye^{\int \frac{2x}{1+x^2} dx} . \sqrt{1 + x^2} = y(1 + x^2)^{\frac{3}{2}}$$

The solution of the given equation is thus

$$y = \frac{1}{\sqrt{1 + x^2}}\left\{ A \tan^{-1} x + B \right\}.$$

Remarks by Mr. J. B. Freeman.

The equation can be written as

$$[(1 + x^2) D + x] \; [(1 + x^2) D + x] \; y = 0$$

and where D stands for the operator $\frac{d}{dx}$, the solution follows easily.

Question 1296.

(V. RAMASWAMI IYER) :—Prove that for every acute-angled triangle ABC, there is an obtuse-angled triangle A′B′C′ and *vice versa*, so that, if O, O′ be their respective ortho-centres

$$OA = O'A', \; OB = O'B', \; OC = O'C'.$$

If the two triangles be so placed that their circum-centres coincide at a point S, then the circum-circle of either will touch the nine-point circle of the other.

Solution (1) *by R. Gopalaswamy,* (2) *by Ram Behari.*

(1) If R and R$'$ be the circum-radii of the two triangles then

$$|R'\cos A'| = R\cos A, \; |R'\cos B'| = R\cos B,$$
$$|R'\cos C'| = R\cos C\;;$$

that is, the absolute values of the cosines of the angles in the triangle ABC are proportionally increased. If B$'$ is obtuse,

$$\frac{\cos A'}{\cos A} = -\frac{\cos B'}{\cos B} = \frac{\cos C'}{\cos C} = \frac{R}{R'} = k, \text{ say.}$$

Now

$$\cos^2 A' + \cos^2 B' + \cos^2 C' + 2\cos A'\cos B'\cos C' - 1 = 0. \quad (1)$$
$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C - 1 = 0. \quad (2)$$

From (1)

$$k^2(\cos^2 A + \cos^2 B + \cos^2 C) - 2k^3\cos A\cos B\cos C - 1 = 0.$$

Using (2) we get

$$k^2(-1 + 2\cos A\cos B\cos C)$$
$$+ 2k^3\cos A\cos B\cos C + 1 = 0;$$

or $$2k^2\cos A\cos B\cos C - k + 1 = 0.$$

Hence $$k = \frac{1 \pm \sqrt{1 - 8\cos A\cos B\cos C}}{4\cos A\cos B\cos C}.$$

But $$SO^2 = R^2 - 8R^2\cos A\cos B\cos C;$$

i.e., $$k = \frac{2R(R \pm SO)}{R^2 - SO^2}; \text{ or, } k = \frac{2R}{R \mp SO}.$$

Let us for the present confine ourselves to the root

$$k = \frac{2R}{R + SO}.$$

Then $$R' = R/k = \tfrac{1}{2}(R + SO). \quad \ldots \quad \ldots (1)$$

Hence if we describe a circle with R$'$ as radius and inscribe in it a triangle with angles A$'$, B$'$, C$'$, so that

$$\cos A' = k\cos A, \cos B' = -k\cos B, \cos C' = k\cos C,$$

we get the triangle required.

To determine SO$'$, we note that if we took our original equations as

$$\frac{\cos A}{\cos A'} = -\frac{\cos B}{\cos B'} = \frac{\cos C}{\cos C'} = \frac{R'}{R} = k',$$

we should have got the quadratic

$$2\,k'^2 \cos A' \cos B' \cos C' - k' + 1 = 0;$$

whose roots are $\dfrac{2\,R'}{R' + SO'}$ and $\dfrac{2\,R'}{R - SO'}$.

The latter is inadmissible as k' should be positive and $R - S'O'$ is negative for an obtuse-angled triangle.

Hence,

$$R = \tfrac{1}{2}\,(R' + SO'.) \qquad \ldots \qquad \ldots \quad (2)$$

We have seen that $R' = \tfrac{1}{2}\,(R + SO)$

Hence $SO' = \tfrac{1}{2}\,(3\,R - SO)$,

$$SO = \tfrac{1}{2}\,(3\,R' - SO');$$

and the relations between the two triangles are thus perfectly reciprocal.

If N' be the nine-point centre of $A'B'C'$,

$$SN' = \tfrac{1}{4}\,(3\,R - SO) = \tfrac{1}{2}\,R - \tfrac{1}{4}\,(R + SO)$$
$$= R - \tfrac{1}{2}\,R'.$$

Hence the nine-point circle of $A'B'C'$ and the circum-circle of ABC have internal contact.

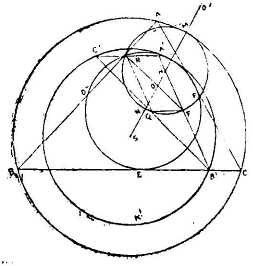
Similarly the reciprocal relation.

[NOTE:—It is easy to see that the second solution

$$k = \frac{2\,R}{R - SO}$$

leads to an obtuse-angled triangle $A''B''C''$, whose vertices are all imaginary. But even in that case, the ortho-centre O'', the nine-point circle, and the circum-circle are all real.

Of the three triangles, Δ, Δ' and Δ'', taking any two, the circum-circle of either touches the nine-point circle of the other, the contact of the circles associated with Δ' and Δ'' being external.]



(2) Let ABC be the acute-angled triangle, O the ortho-centre, and S the circum-centre. Bisect SO at N, then N is the centre of the nine-point circle, and the radius of the nine-point circle is half the radius of the circum-circle of ABC.

Draw the nine-point circle DEF, where D, E, F are the mid.-points of AB, BC, CA respectively. Produce SNO to O' meeting the circle DEF in L, and the circle ABC in M. Then

since all the lines drawn from the ortho-centre of a triangle to the circumference of its circum-circle are bisected by its nine-point circle, we get

$$OL = LM.$$

Make $MO' = ML$. Bisect SO' at N'. With N' as centre and $\frac{1}{2}$ SL as radius describe a circle cutting the three circles drawn with S as centre, and SD, SE, SF as radii at P, Q, R respectively. Also with S as centre and SL as radius draw the circle $A'B'K'C'$. Join PQ, QR, RP. Through Q draw a line $B'QC'$ parallel to PR; through R draw $C'RA'$ parallel to PQ; through P draw $A'PB'$ parallel to QR.

Then $A'B'C'$ is a triangle similar to the triangle PQR; it has the circle $A'B'K'C'$ for its circum-circle, and is such that P, Q, R are the middle points of $A'B'$, $B'C'$, $C'A'$ respectively.

Now in the circle $A'B'K'C'$, S is the centre and P, Q, R are the mid-points of the chords $A'B'$, $B'C'$, $C'A'$. Therefore SP, SQ, SR are perpendicular to $A'B'$, $B'C'$, $C'A'$, respectively.

Hence S is the circum-centre of $A'B'C'$.

The circle PQR is the nine-point circle of the triangle $A'B'C'$. Now since N', the centre of the nine-point circle of the triangle $A'B'C'$, must bisect the line joining S, its circum-centre, to its ortho-centre, and also since $SN' = \frac{1}{2} SO'$, it follows that O' is the ortho-centre of the triangle $A'B'C'$.

Again since the distance of each vertex of a triangle from its ortho-centre is twice the perpendicular distance of the circum-centre from the side opposite to that vertex, we get

$$O'A' = 2SQ,\ O'B' = 2SR,\ O'C' = 2SP.$$

But $SQ = SE = \frac{1}{2} OA$; $SR = SF = \frac{1}{2} OB$; $SP = SD = \frac{1}{2} OC$.

$$\therefore \quad O'A' = OA,\ O'B' = OB,\ O'C' = OC.$$

Also since the centres of the two circles $A'B'K'C'$ (the circum-circle of the triangle $A'B'C'$) and DEF (the nine-point circle of the triangle ABC), lie on the straight line SNL, they touch each other at L.

Similarly since the two circles ABC and PQR have their centres on the straight line $SN'M$, they touch each other at M.

Again the radius of the nine-point circle PQR being half of the radius of the circum-circle $A'B'K'C'$ is obviously less than half the radius

of the circle ABC, (*i.e.*, the diameter through M of the circle PQR is less than MS, the radius of the circle ABC). Therefore the circle PQR which touches the circle ABC at M, cuts the three circles drawn with S as centre and SD, SE, SF as radii, at points on the nearer side of S. The triangle A′B′C is therefore situated in a segment smaller than a semi-circle, and hence the angle B′A′C′ which is opposite to the longest side B′C′, is an obtuse angle.

Question 1300.

(M. G. IMAMDAR) :—Prove that the area of a triangle can be got from the formula

$$\Delta = 4\{(\sigma - p)(\sigma - q)(\sigma - r)(\sigma - R)\}^{\frac{1}{2}},$$

where $2\sigma = p + q + r + 2R$ and p, q, r, are the perpendiculars from the circum-centre on the sides of the triangle and R is the circum-radius.

Solution (1) *by K. O. Shah,* (2) *by K. Satyanarayana,*
(3) *by R. Srinivasan, Ganesan, I. Totadri Iyengar, M. K. Kewalramani, and P. Seetarama Rao.*

(1) Let ρ, r_1, r_2, r_3 be the in-radius and ex-radii of the triangle. If O is the circum-centre, A_1 the mid.-point of BC and TOA_1T_1 the diameter of the circum-circle, then it can be easily proved that

$$2TA_1 = r_2 + r_3.$$

$$\therefore \quad 2p = 2OA_1 = 2TA_1 - 2TO$$
$$= r_2 + r_3 - 2R.$$

Similarly $\quad 2q = r_3 + r_1 - 2R,$

$$2r = r_1 + r_2 - 2R.$$

$$\therefore \quad 4\sigma = 2(r_1 + r_2 + r_3) - 4R.$$

$$\therefore \quad 2\sigma = r_1 + r_2 + r_3 - 2R.$$

Hence $2(\sigma - p) = r_1$, $2(\sigma - q) = r_2$, $2(\sigma - r) = r_3$;

and $\quad 2(\sigma - R) = r_1 + r_2 + r_3 - 4R$

$$= \rho, \text{ for } r_1 + r_2 + r_3 - \rho = 4R.$$

$$\therefore \quad 16(\sigma - p)(\sigma - q)(\sigma - r)(\sigma - R) = r_1 r_2 r_3 \rho = \Delta^2.$$

$$\therefore \quad \Delta = 4\{(\sigma - p)(\sigma - q)(\sigma - r)(\sigma - R)\}^{\frac{1}{2}}.$$

(2) ABC is an acute-angled triangle in which D, E, F are the mid.-points of the sides and S is the circum-centre. Draw DF′ equal and parallel to SF.

Then, since DE, S′F′ are respectively parallel to AB, AC, EDF′, ESF′ are right angles. Hence ESDF′ is cyclic.

Also, from the parallelogram ASF′E,

$$EF' = SA = R$$

and the quadrilateral ESDF′ has its sides equal to p, q, r, R.

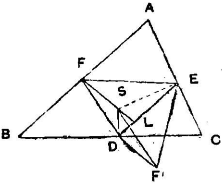
Hence its area $= \sqrt{(\sigma - p)(\sigma - q)(\sigma - r)(\sigma - R)}$.

But if SF cuts DE in L,

$$\Delta\, SFD + \Delta\, SFE = \tfrac{1}{2}\, SF\,(DL + LE) = \Delta\, DEF',$$

$$\therefore \qquad \text{area } ESDF' = \Delta\, DEF = \tfrac{1}{4}\,\Delta\, ABC.$$

Hence the result.



(3) It is evident that

$$p = R\cos A,\ q = R\cos B \text{ and } r = R\cos C.$$

$$\sigma - p = \tfrac{1}{2}R\,(1 + \cos B + \cos C - \cos A)$$

$$= 2R \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

$$\sigma - R = \tfrac{1}{2}R\,(\cos A + \cos B + \cos C - 1)$$

$$= 2R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$$\therefore \qquad 4\sqrt{\Pi(\sigma - p)} = 4\sqrt{\left\{16R^4.\,\Pi\left(\sin^2\frac{A}{2}\cos^2\frac{A}{2}\right)\right\}}$$

$$= 2R^2 \sin A \sin B \sin C.$$

Questions for Solution.

1367. (S. D. CHOWLA):—Prove that

$$e^{-\beta^2} - e^{-(5\beta)^2} - e^{-(7\beta)^2} + e^{-(11\beta)^2} + e^{-(13\beta)^2}$$
$$- - + + \ldots\ldots$$
$$= \frac{\sqrt{\pi}}{2\sqrt{3}\beta}\left\{ e^{-\left(\frac{\pi}{12\beta}\right)^2} - e^{-\left(\frac{5\pi}{12\beta}\right)^2} - e^{-\left(\frac{7\pi}{12\beta}\right)^2} + + - - \right\}.$$

1368. (R. GOPALASWAMI):—Given four fixed points A, B, C, D, show that there are, in general, two conics through them whose asymptotes include a given angle θ and that the conics are identical for two particular values of θ, one of which is always equal to $\frac{1}{2}\pi$. If these conics be S_1 and S_2, then

(*a*) S_1 and S_2 cut the circle through any three of the four points A, B, C, D at diametrically opposite points.

(*b*) The centres of the conics S_1 and S_2 are the extremities of a diameter of the nine-point conic of ABCD.

1369. (R. GOPALASWAMI):—A_1, B_1, C_1, D_1 are the isogonal conjugates of each of the points A, B, C, D, with respect to the triangles formed by the other three. If for these triangles

$$\{S_1, \rho_1\}\ \{S_2, \rho_2\}\ \{S_3, \rho_3\}\ \{S_4, \rho_4\}$$

are the circum-centres and circum-radii respectively, then

$$\frac{A_1S_1}{\rho_1} = \frac{B_1S_2}{\rho_2} = \frac{C_1S_3}{\rho_3} = \frac{D_1S_4}{\rho_4}.$$

1370. (V. THIRUVENKATACHARYA):—If $\alpha\beta = \pi^2$, show that

(*a*) $$\frac{\pi}{2}\cot\sqrt{\alpha t}\coth\sqrt{\beta t} = \frac{1}{2t} + \frac{\alpha\coth\alpha}{t+\alpha} + \frac{2\alpha\coth 2\alpha}{t+4\alpha} + \ldots\ldots$$
$$+ \frac{\beta\coth\beta}{t-\beta} + \frac{2\beta\coth 2\beta}{t-4\beta} + \ldots\ldots$$

(b) $$\frac{\pi^2}{4}\operatorname{cosec}^2\sqrt{\alpha t}\operatorname{cosech}^2\sqrt{\beta t} = \frac{1}{4t^2} + \frac{\alpha^2\operatorname{cosech}^2\alpha}{(t+\alpha)^2} + \frac{4\alpha^2\operatorname{cosech}^2 2\alpha}{(t+4\alpha)^2} + \ldots\ldots + \frac{\beta^2\operatorname{cosech}^2\beta}{(t-\beta)^2} + \frac{4\beta^2\operatorname{cosech}^2 2\beta}{(t-4\beta)^2} + \ldots\ldots$$

1371. (B. B. BAGI) :—If at the middle points of an acute-angled triangle ABC, perpendiculars to the sides are erected of lengths

$$\tfrac{1}{2}r_1,\ \tfrac{1}{2}r_2,\ \tfrac{1}{2}r_3.$$

all drawn outwards, then the triangle whose vertices are the extremities of these perpendiculars is similar to the triangle of ex-centres.

1372. (K. SATYANARAYANA) :—

(i) If a be any given positive quantity >1, show that as p increases from $-\infty$ to $+\infty$,

$$\left[\frac{a^p+1}{2}\right]^{\frac{1}{p}} \text{ increases crossing } \sqrt{a} \text{ as } p \text{ crosses } 0.$$

(ii) Find the limit of $\left[\frac{a^p+1}{2}\right]^{\frac{1}{p}}$ as

(a) p tends to zero through positive or negative values,

(b) p tends to $\pm\infty$.

1373. (MARTYN THOMAS) :—The equation to a system of curves, in bi-polar co-ordinates, is

$$r^2 - r_1^2 - 2ar + a^2 = 0.$$

Show that the equation to the orthogonal system, in bi-angular co-ordinates, is

$$\tan^2\theta - 2k\tan\theta\operatorname{cosec}2\theta_1 + k^2 = 0.$$

1374. (HANS R. GUPTA) :—If $\frac{p_n}{q_n}$ denote the nth convergent of the C. F.

$$\cos\theta - \frac{1}{2\cos\theta -}\ \frac{1}{2\cos\theta -}\ \frac{1}{2\cos\theta -}\ \ldots\ldots,$$

show that $p_n = \cos n\theta$.

Find a similar C. F. for which $p_n = \sin n\theta$.

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LIST OF JOURNALS RECEIVED

(From 1st December 1924 to 7th January 1925).

1 Philosophical Magazine, Vol. 48, Nos. 287, 288.

2 The American Mathematical Monthly, Vol. 31, No. 8 (two copies).

3 Annals of Mathematics, Vol. 25, Nos. 3, 4.

4 Proceedings of the London Mathematical Society, Vol. 23, Part 5 (three copies)

5 Popular Astronomy, Vol. 32, No. 8.

6 Nature, Vol. 114, Nos. 2865, 2866, 2867, 2868, 2869, 2870, 2871, 2872 and 2873.

7 Bulletin of the American Mathematical Society, Vol. 30, No. 8 (two copies).

8 Transactions of the American Mathematical Society, Vol. 24, No. 2; Vol. 26, No. 4.

9 Monthly Notices of the Royal Astronomical Society, Vol. 84, No. 9.

10 The Astrophysical Journal, Vol. 60, No. 3.

11 Proceedings of the Royal Society, Series A, Vol. 106, No. 739.

12 Philosophical Transactions of the Royal Society, Series A, Vol. 224, Nos. A 622, 623.

13 Transactions of the Cambridge Philosophical Society, Vol. 23, Nos. 1, 2, 3, 4, 5 and 6.

14 Proceedings of the Cambridge Philosophical Society, Vol. 22, Parts 1, 2 and 3.

15 Annals de la Faculte des Sciences de Toulouse, Tome XV, Anne 1923, (Troisieme Series).

16 Bulletin des Sciences Mathematiques, Vol. 48, October issue 1924. (Deuxieme Series).

17 Proceedings of the Royal Society, Vol. 106, No. 740, (Series A).

18 Mathematical Gazette, Vol. XII, No. 173 (two copies).

19 Annals de l'Ecole Normale Superieure, Tome 41, No. 8.

20 Memories de La Classe Des Sciences Physique et Mathematiques, Tome I Fascicules, 1, 2 and 3.

21 Bulletin de La Classe Des Sciences Physique et Mathematiques, Tome I, Fascicules 1 & 2.

} From Académie des Sciences de l'Oukraine.

22 Encyklopädie Des Mathematischen Wissenschaften.

(All the bands and teils, that are published up to this time, are received).

'On a Table for calculating Eulerian Numbers based on a New Method,' by C. Krishnamachary and M. Bhimasena Rao.

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